tions, and projective geometry. An almost equally important set of first courses in these three domains would be: theory of discrete groups, functions of a complex variable, and differential geometry. A third important set is higher algebra, functions of a real variable, and algebraic geometry. After completing one or two such general courses in each of these great domains, the student will probably have selected his special subject for thorough study and he may then wisely select more special courses. The number of such special courses that may be open to him will depend more upon the size of the faculty than upon the nature of the subjects. In the great encyclopedia the subject matter of each of the first two grand divisions of mathematics is presented under about thirty general headings, while that of geometry is placed under about forty such headings. As each of these headings is abundantly extensive for a course of lectures, we have here suitable names and material for about one hundred courses, without going outside of the field of pure mathematics and without general considerations of historic and pedagogic subjects.

From this great abundance of material and the tendency to give courses on very special subjects it is evident that the student who would acquire a comprehensive knowledge of mathematics has to do a large amount of reading on subjects which may not be closely related to the courses selected by him. In many of our universities this is especially true as regards the history of mathematics. If mathematicians wish to keep in contact with each other end especially with the larger circle of thinkers who are interested in the general intellectual development, they need a comprehensive view of the history of the various fundamental concepts of mathematics. Such a view is not generally acquired without much effort and the serious student will find that he can only secure a fairly satisfactory knowledge along such general lines by a persistent and wise use ot the moments which are not required for his regular work. Comprehensive historic knowledge will also tend to calmer judgment as regards the ephemeral waves of superficial interest in particular subjects.

We have thus far spoken only of pure mathematics. This is naturally the field of earliest interest as one cannot apply any unknown mathematics, and the applications are frequently more difficult than the considerations as regard ideal conditions of a simple nature. The young mathematicians should however bear in mind that not only ought mathematics be developed harmoniously as a whole, but this development should go hand in hand with the harmonious development of its applications. Sometimes these applications mean only a change of language, and there is no distinct boundary between pure and applied mathematics. Nevertheless, it is of the utmost importance that the mathematician should be able to express his results in the most effective and in the most useful language, and this demands a knowledge of the fundamental laws of the subjects to which mathematical thought may be applied. The wider and more thorough this knowledge the more

probable it will be that the results may be put into the most helpful form. It is in the border land between mathematics and various other sciences where one may reasonably look for the most useful developments, and next to these come the border lands between the various mathematical subjects themselves.

One striking feature of the mathematics beyond the calculus is that, as a rule, it is not arranged in such a carefully graded form as the earlier mathematics. If one considers how few new concepts enter into a course in elementary calculus, or a course in analytic geometry, and how much time and space is used in viewing these concepts from numerous points and how minute most of the thought steps are, one may wonder whether all this was really necessary. As a matter of fact, there is a great waste of time for the brightest student, but so many students of mediocre ability have to get over this ground that these details have been generally adopted.

On the other hand the average ability of the students who go beyond the elementary course in calculus is so very much higher that the thought steps are generally much longer. In fact, they are frequently of such lengths as to discourage the student. This discouragement generally lasts only a short time, and it is frequently replaced by keen enthusiasm, when the student begins to appreciate great thoughts rather than details of calculation, and when he learns to select view-points according to his own taste instead of slavishly following directions. It is true that this requires more time but it also brings better results and prepares the way for independent thought. As all mathematics consists in going from one thought to another near by and repeating this process, it is clear that the power of independent thought is the mathematician's El Dorado.

There is a tendency to speak of the relative importance of work in the different fields of mathematics. In this respect mathematics has much in common with mining. Some of the most enthusiastic reports are based on surface indications or upon developments which are totally inadequate to justify the reports. It is also true that some prospectors seem to lose the ability of passing calm judgments or speaking cautiously. The young mathematician should bear in mind that some of the largest fortunes have been acquired by working low grade ore by improved processes, and that the usual fate of the superficial prospector is not inspiring. In each of the three great grand divisions of mathematics there is in sight an inexhaustible supply of ore of various grades. Some prospecting work in these fields is, however, also highly desirable.

As great theories can be developed only by collaboration it is very important that mathematicians should publish their most useful results from time to time and that they should put them into best possible form. The young mathematician should, however, be very sure of the importance and newness of his results before he offers them for publication. In De Morgan's Budget of Paradoxes, page 4, there occurs the following good advice along this line:

"Most persons must, or at least will, like the lady in Cadogan Place, form and express an immense variety of opinions on an immense variety of subjects; and all persons must be their own guides in many things. So far all is well. But there are many who, in carrying the expression of their own opinions beyond the usual tone of private conversation, whether they go no further than attempts at oral proselytism, or whether they commit themselves to press, do not reflect that they have ceased to stand upon the ground upon which their process is defensible. Aspiring to lead *others*, they have never given themselves the fair chance of being led by *other* others into something better than they can start for themselves; and that they should first do this is what both these classes of *others* have a fair right to expect."

The student should bear in mind that it is necessary for the instructor to express opinions upon an immense variety of subjects and that some instructors seem to think it also necessary to assume the attitude that they know more about every subject under discussion than all the rest of the world put together. Students should not take such attitudes too seriously even if there may be a tendency among their fellow students to accept instructors at the instructors' estimates of their own abilities. Few habits are so harmful to a student as that of a slavish or even a worshipful attitude towards his instructors. On the other hand, he should seek information from many sources outside of the lecture room. If he does this he will become a more independent thinker and lay a much broader foundation for later development.

It should also not be assumed that all the mathematical talent of the world is concentrated in one locality. In mathematics we are still an idolatrous and thoughtless people, and one of the greatest evils we have to contend against is the worship of idols. Let all preach the gospel of only one *mathematics* and that all mathematical worship should be before his throne only. What one individual may do is insignificant when compared with the total development. Much of the time employed in mathematical pilgrimages to Mecca could be more wisely employed. These pilgrimages, however, have spread the contagion of enthusiasm and in this way they have done a great deal of good. The improved facilities as regards communication, especially by means of the mathematical periodicals, make every good library a suitable place for productive mathematical activity; and sound scholarly work should meet, and generally does meet, with a hearty reception on the part of the greatest mathematicians irrespective of the place where the work was accomplished.

In a very general way it may be said that arithmetic and algebra form the basal subjects of mathematics, that analysis is dependent upon arithmetic and algebra, and that geometry depends upon both of the other two grand divisions. The interdependence of these grand divisions is, however, becoming more and more pronounced, and there is a considerable part of geometry which has only a little in common with the other two divisions.

Hence a specialist in geometry may confine himself within just as narrow limits as a specialist in arithmetic and algebra may do. Since analysis is only algebra grown big, it is clear that it is more directly dependent upon the preceding grand division than geometry is.

As mathemathetics largely consists in going from one concept to a related concept, and as concepts which involve the most extensive intimate relations with others are the most helpful, it is clear that it is just as important at times to be able to overlook details as it is at other times to obtain interesting illuminating results from a study of details. For instance, Hamilton, in his lectures on quaternions, makes a distinction between operator and operandus which is unnecessary for applications and is also a hindrance to clear exposition.* One of the great advantages of abstract group theory rests upon overlooking the distinction between operator and operand in many applications. The distinction between a system of conjugate substitutions and a group of permutations as used by Cauchy and Serret is also of doubtful value. The mathematician must be able to leap from mountain top to mountain top as well as to dig out the gold from old river beds concealed by the work of geologic ages.

A CURIOUS MECHANICAL PARADOX.

By EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

Whether a paradox appears merely as a pest depends largely on the point of view. If the paradox lies in some well known and thoroughly accredited discipline such as elementary algebra, geometry, or mechanics, it is certainly a nuisance except in so far as it may be pedagogically instructive for the purpose of reinforcing, by its solution, the very principles which it would upset. To this class belong the demonstrations that $2=4$, usually dependent on a division by 0 or a carelessly placed sign in the extraction of a square root, and the proof that all triangles are isosceles or all angles right angles, dependent on incorrectly drawn figures, and finally the proposition that a rolling wheel must be lighter than one at rest owing to the resultant centrifugal force attributable to the rotation of the wheel about its instantaneous center. Such paradoxes are even a hindrance pedagogically rather than a help unless the fallacy involved can be made much clearer than the fallacious demonstration—a thing almost impossible to accomplish unless the student is so well gounded in the fundamentals of the science that he would not himself fall into like errors. On the other hand, when the paradox arises in a field which is not yet familiar even to specialists in it, a thorough

*E. Study, *Encyklopaedie der Mathematischen Wissenschaften*, vol. I, page 159.

examination of the paradox may be of first rate scientific importance. The attention which eminent men have paid to the contradictions developable in the theory of transfinite numbers and in the early discussions of the second law of thermodynamics might be mentioned as illustrations. It is doubtful if all the paradoxes connected with kinetic theory are yet cleared away from the statistical study of the interactions of matter and ether; and from the point of view of the new theory of relativity the Lorentz shortening of a body in its line of motion may easily take on the aspect of a paradox.

A curious and withal instructive example of the former kind of paradox has just been brought to my attention and seems worthy of solution not only for those who have been puzzled by it but for those who may never have thought of it. Here it is — problem, paradox, and solution.

PROBLEM. Suppose a tank of water with an efflux pipe, which terminates vertically, is resting on a frictionless plane when the water begins to flow. What is the motion of the system?

To make the problem as specific as possible and at the same time to remove any unnecessary lack of symmetry, let it be assumed that the efflux pipe 1234 starts from the center of the bottom of the tank, runs through a horizontal distance L and ends with a vertical section as in the figure. It is this vertical ending which introduces the paradox.

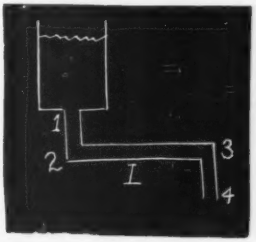


Fig. 1.

PARADOX. Let the water begin to flow out. As there are no horizontal external forces acting on the system, the center of gravity of the whole system must remain in the same vertical line. Hence the tank must slide to the left as the water is carried to the right. But as the tank slides to the left the water which leaves at 4 has a horizontal component toward the left and again, as there is no external horizontal force, the water will keep on indefinitely moving toward the left which is the same direction that the tank moves. Hence the center of gravity of the system will ultimately be found far to the left of its original position instead of in the same vertical line.

SOLUTION. The tank does start moving toward the left and a certain amount of water will move indefinitely to the left. But the tank will ultimately reverse its direction and move to the right and some of the water will move indefinitely to the right. And at all times the center of gravity of the system will be in the same vertical line and the system as a whole will never have any horizontal momentum.

That this is necessarily the solution is of course apparent as soon as one fixes his attention on the laws of momentum and abandons the attempt to follow intuitively and in detail the actual motion of the tank — but there would not be many paradoxes if one could only fix his attention rightly at the start. The really interesting thing about this particular paradox, how-

ever, is not its qualitative solution but the quantitive solution that follows immediately from applying the laws of mechanics to the system. Let v be the velocity of the water flowing in the tube 1234 and measured relatively to the tube. Let u be the velocity of the tank measured positively to the right. Let A be the cross-section and L the length of the tube 23. Let W be the amount of water in the tank and tubing at any instant and M the mass of the tank and tube. Then

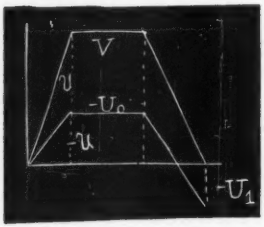


Fig. 2.

$$(M+W)u+AL(v-u)=(M+W-AL)u+ALv$$

is the total momentum at any time of the whole containing vessel and the water contained therein but not of the water which has flowed out; the total momentum of the water which has escaped is the negative of this. The rate at which water is flowing out is Av and as the velocity of tank is u, the rate of loss of momentum from the tank and tube is Avu. Hence

$$\frac{d}{dt}[(M+W-AL)u+ALv]=-Avu,$$

or

$$(M+W-AL)\frac{du}{dt}+AL\frac{dv}{dt}=0, \qquad (1)$$

since dW/dt, the rate of loss of water, is $-Av$. One may also write

$$(M+W)\frac{du}{dt}+AL\frac{dv'}{dt}=0, \qquad (1')$$

where $v'=v-u$ is the actual velocity of the water in the tube.

To tell just how the water would actually flow out would probably be a hydrodynamical problem incapable of solution in terms of elementary functions; but if any one of a number of reasonable assumptions be made as to the rate of change of v or v', the equations (1) or (1') may be used to solve the motion of the tank. For instance, suppose that, when the efflux pipe is opened, the velocity v rises rapidly from 0 to a steady value V which is maintained for a while and then falls rapidly to 0 as the pipe is closed. As the rise of v is rapid, the coefficient $M+W-AL$ may be treated as constant during the rise. Hence $-u$ will rise from 0 to $-U_0$ where

$$-U_0=\frac{AL}{M+W_0-AL}V \text{ and } W_0 \text{ is the initial value of } W.$$

Then the constant value U_0 will be maintained so long as V is maintained. Finally when v returns to 0 the coefficient $M+W-AL$ may again be regarded as constant and it is seen that u comes to the value

$$U_1 = -\frac{AL}{M+W_0-AL}V+\frac{AL}{M+W_1-AL}V, \text{ where } W_1 \text{ is the final } W.$$

This value of u is not 0 but positive, and indeed is necessarily so in order to counterbalance the negative momentum of the water which has escaped. The figure represents the variation of v and $-u$.

Other reasonable assumptions might be made, for instance Torricelli's law of efflux. Moreover the actual displacement of the tank could be obtained by integrating u. But sufficient has been said to show how readily the problem may be treated as far as its purely mechanical side is concerned. As it is none too easy to find really good and yet essentially different problems which require for their solution merely the fundamental principles of momentum, energy, and moment of momentum, it may not be amiss to point out that the above problem may be relieved of its hydrodynamical difficulties but otherwise preserved intact by considering a small number of beads on a wire lying in a vertical plane and consisting of a series of segments at different inclinations to the horizontal.

THE THEORY OF INVERSION AND THE QUADRATIC RECIPROCAL TRANSFORMATION.

By D. N. LEHMER, University of California.

1. The following discussion exhibits the theory of inversion in its proper light as a quadratic reciprocal transformation, and makes clear why inversion throws circles into circles but generally a curve of degree n into a curve of degree $2n$.

2. Consider a conic K and a point M. A correspondence between the points of the plane may now be set up as follows: To any point P make correspond the point P' of intersection of the line MP with the polar of P with respect to K. The point P' goes by this process into the point P again by the fundamental theorem in the theory of poles and polars: If the polar of P passes through P', the polar of P' passes through P.

If the point P moves along a straight line the point P' moves along a conic.

For P' is the locus of intersection of two projective pencils with centers at M and at the pole of the line upon which P moves. This conic thus

passes through a point M. A slight consideration will show also that it passes through the points A and B where the tangents from M to K touch K. For let P be on one of these tangents, then the line PM meets the polar of P in one of the points of tangency. Further, if P lies on the line AB the corresponding point P' is at M. More generally, we shall prove that

The transformation above described tranforms a curve of degree n into a curve of degree $2n$ which passes n times through the points A, B, and M.

For let C be a curve of degree n. It transforms into a curve C' of degree which it is proposed to determine. Cut across C' by a straight line a. Transform now again and C' goes back into C and the line a into a conic a'. The $2n$ points of intersection of a' with C correspond to the points of intersection of the line a and C'. Further, the curve C cuts the line AB in n points. The curve C' then passes n times through M. Similarly for points A and B.

The transformation is a little different from the ordinary transformation by which lines go into conics in that ordinarily points on a side of the singular triangle go into the *opposite* vertex. Here, while this is the case for the side AB, it is not the case for the other two sides, MA and MB.

The usual considerations apply for curves that go through one or more of the vertices A, B, and M of the singular triangle. Thus a conic generally goes into a quartic with three double points A, B, and M. If the conic goes through M the quartic degenerates into a straight line AB and a cubic through A, B and M with double points at M. If the conic goes through the points A and B the quartic becomes a conic and two straight lines. The two lines are MA and MB, the conic goes through A and B. Finally, if the conic goes through M, A, and B it transforms into a quartic made up of four straight lines one of which is the line that corresponds to that conic by the transformation; the other three being the sides of the singular triangle.

Consider now the special case where K is a circle and M is the center of that circle. Everything noted above still holds. If P moves along a line the point P' moves along a conic through M, A, and B. It is now seen that the points A and B are the points, usually denoted by I and J, where the circle K meets the polar of M, that is the line at infinity. If we define a circle as a conic through I and J, we have a transformation that throws a straight line into a circle. Moreover, a circle by this transformation goes into a circle, for as we have seen a conic through A and B goes into a conic through A and B. A conic generally goes into a quartic with double point at the center of K and double points at the circular points at infinity.

The metrical properties of the theory of inversion are now easily built up. It is not necessary to indicate the process further.

ON THE SOLUTION OF A SYSTEM OF LINEAR EQUATIONS.

By G. A. MILLER, University of Illinois.

A general system of m linear equations in n unknowns may be denoted as follows:

$$T=\begin{cases} a_{11}\,x_1+a_{12}\,x_2+\ .\ .\ .\ +a_{1n}\,x_n+k_1=0, \\ a_{21}\,x_2+a_{22}\,x_3+\ .\ .\ .\ +a_{2n}\,x_n+k_2=0, \\ \quad .\qquad .\qquad .\qquad .\qquad .\qquad . \\ \quad .\qquad .\qquad .\qquad .\qquad .\qquad . \\ a_{m1}\,x_1+a_{m2}\,x_2+\ .\ .\ .\ +a_{mn}\,x_n+k_m=0. \end{cases}$$

In the study of the solutions of this system the following two matrices are of especial importance.

$$A=\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ \quad . & \quad . & & & & \quad . \\ \quad . & \quad . & & & & \quad . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{vmatrix}, \qquad B=\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} & k_1 \\ a_{21} & a_{22} & . & . & . & a_{2n} & k_2 \\ \quad . & \quad . & & & & \quad . & \quad . \\ \quad . & \quad . & & & & \quad . & \quad . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} & k_m \end{vmatrix}.$$

Capelli exhibited the pedagogic advantage in employing, in the study of the system T, the concept of rank of these matrices, which are known, respectively, as the matrix and the augmented matrix of the system. The rank of a matrix is, according to Frobenius, the order of the largest non-vanishing determinant formed by elements of the matrix in order. The theorem proved by Capelli may be stated as follows: The necessary and sufficient condition that the system T is solvable is that the rank of the matrix of T is equal to the rank of its augmented matrix.* By solvable is meant that a set of n *finite* values may be assigned to the unknowns x_1, x_2, ..., x_n, so that each of the equations in T is satisfied.

When the system T is solvable it is also said to be consistent or compatible. If the system T is consistent and of rank r we may assign arbitrary values to at least one set of $n-r$ unknowns in this system so that after this is done it is possible to solve the resulting system. To each such $n-r$ arbitrary values there will correspond a single set of values for the remaining r unknowns, provided the $n-r$ unknowns to which arbitrary values were assigned were so selected that the rank of the matrix of the system is not diminished by omitting the coefficients of these $n-r$ unknowns from the matrix. As this interesting theory is so clearly presented in Bôcher's

*Capelli, *Rivista di Matematica*, Vol. 2 (1892), page 54. Capelli used the term characteristic instead of rank. The former of these terms is frequently employed in France and Italy. Cf. Pincherle, *Lezioni di algebra complementare*, 1909, page 92.

Introduction to Higher Algebra it does not appear necessary to enter into greater details here.

The main object of the present note is to consider the question: when can a given unknown in a consistent system of equations have only one value. To make this question perfectly clear we may consider the following system of three equations in three unknowns:

$$\begin{aligned} 2x-y+2z&=8,\\ 4x-2y-z&=-4,\\ 6x-3y+z&=-4. \end{aligned}$$

It is evident that the rank of the matrix of this system is equal to the rank of the augmented matrix, each being 2. For every arbitrary value of x there is one and only one pair of values for y and z such that each of these three equations is satisfied. For instance, when $x=0$, y and z must have the value 0, 4 respectively; and when $x=1$ the values of y and z are 2, 4 respectively. Similarly, there is one and only one pair of values of x and z for every arbitrary value of y. On the contrary, z must always have the same value, namely, 4. We have therefore a system here in which the unknown z can have only one value although the system has an infinite number of solutions.

We are now in position to understand the following theorem:

The necessary and sufficient condition that a given unknown in a consistent system can have only one value is that the rank of the matrix of the system is diminished by omitting the coefficients of this unknown from this matrix.

That this condition is necessary follows directly from the general theory mentioned above; for, if the rank of this matrix were not diminished by omitting the given coefficients we could assign an arbitrary value to this unknown and solve the resulting system. Hence it remains only to prove that the given condition is also sufficient.

Suppose that the system T is consistent and that the rank of its matrix A is r, but that the rank of the matrix A' obtained by omitting the coefficients of x_a from A is less than r. The rank of A' must therefore be $r-1$ as the co-factor of at least one of these coefficients cannot vanish in a non-vanishing determinant of order r contained in A. As the matrix A' is of rank $r-1$ each of its rows can be expressed as a linear function of $r-1$ of these rows.* Hence we may replace the system T by a new system T' having the following form:

$$\begin{aligned} a_{1a}x_a + l_1 + k_1 &= 0,\\ a_{2a}x_a + l_2 + k_2 &= 0,\\ \cdot\quad &\cdot\quad \cdot\quad \cdot\quad \cdot \end{aligned}$$

*Cf. Bocher, *Introduction to Higher Algebra*, page 36.

$$
\begin{aligned}
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&a_{r-1a}\, x_a + l_{r-1} + k_{r-1} = 0, \\
&a_{ra}\, x_a + \phi_0(l_1, l_2, \ldots, l_{r-1}) + k_r = 0, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&a_{ma}\, x_a + \phi_{m-r}(l_1, l_2, \ldots, l_{r-1}) + k_m = 0,
\end{aligned}
$$

where $\phi_0, \ldots, \phi_{m-2}$ are the linear functions of $l_1, l_2, \ldots, l_{r-1}$.

The system T' is consistent since T is consistent and it must be of rank r since T has this rank. Hence it results that one and only one set of values for $x_a, l_1, \ldots, l_{r-1}$ will satisfy the first r equations. That is, x_a can have only one value in system T'. It can therefore have only one value in system T and the theorem in question has been established. From this theorem it results that each unknown in system T has either only one value or it has an infinite number of values whenever this system is consistent. In particular, a system of linear equations has always an infinite number of distinct solutions whenever it has more than one solution.

From the above it is evident that the language as regards solvability of a system of linear equations becomes much more concise by means of the concept of rank. Although this concept is comparatively new in mathematics it is of such fundamental importance that it should occupy a more prominent place in the courses in advanced algebra. It should be remembered that a thing can only appear simple when we can see clearly through it. In particular, the theory of linear equations appears simple only after an exhaustive study by means of such a powerful instrument as the concept of rank.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

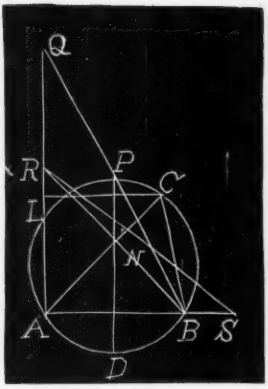
GEOMETRY.

363. Proposed by G. I. HOPKINS, Manchester, N. H.

Construct the triangle, having given, base, vertical angle, and difference between altitude and sum of the other two sides.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In what follows we assume that $AB=a$=the given base; $\angle ACB=\angle C$=given vertical angle; p=difference of altitude and sum of other sides.



We further assume that $p>a<a\cot\frac{1}{2}C$. The solution would be just as simple with any other possible assumption.

On AB describe the segment ACB containing the given angle C. Draw the diameter PD perpendicular to AB. Also draw AQ perpendicular to AB. Draw BP meeting AQ in Q. With B as a center and a radius equal to p, describe an arc cutting AQ in R. With R as a center and a radius equal to $AQ-p$, describe an arc cutting AB produced in S. On SR measure off $SN=SA$. Then $RN=$altitude required. Take $AL=RN$, and draw CL parallel to AB, cutting the circle in C. Draw AC, BC. Then ACB is the required triangle. Let x, y, z be the sides BC, AC, and the altitude. Then $xy\sin C=az$, $x+y-z=p$, $a^2=x^2+y^2-2xy\cos C$.

$\therefore a^2+2xy(1+\cos C)=(p+z)^2$.

$\therefore z^2-2(a\cot\frac{1}{2}C-p)z=a^2-p^2$.

$\therefore z=a\cot\frac{1}{2}C-p-\sqrt{[(a\cot\frac{1}{2}C-p)^2-(p^2-a^2)]}$.

$\angle AQB=\frac{1}{2}C$, $AQ=a\cot\frac{1}{2}C$, $RS=a\cot\frac{1}{2}C-p$, $AR=\sqrt{[p^2-a^2]}$, $AS=\sqrt{[(a\cot\frac{1}{2}C-p)^2-(p^2-a^2)]}$.

$\therefore RN=z$. $\therefore$ The triangle ACB contains all the required parts.

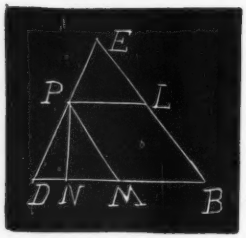
Also solved by J. Scheffer.

364. Proposed by R. C. ARCHIBALD, Providence, R. I.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. ["Euclidean constructions" are particularly desired.]

Solution by C. N. SCHMALL, New York City, and J. SCHEFFER, A. M., Hagerstown, Md.

This construction cannot be effected by Euclidean geometry. This will be evident from the algebraic analysis of the conditions.



Let $MBLP$ be the given rhombus, and P the given corner. Also let DE be the required line, so that $DE=l=$given length. On AB drop the perpendicular PN. Now let $MB=BL=a$, $MN=c$, $DM=x$. Then we have, in $\triangle DMP$,

$$DP^2=DM^2+MP^2-2DM.MN=x^2+a^2-2cx.$$

Also, $DM^2:DP^2=DB^2:DE^2$;

that is, $x^2:(x^2+a^2-2cx)=(x+a)^2:d^2$.

$$\therefore d^2=\frac{(x+a)^2(x^2-2cx+a^2)}{x^2}=(x+a)^2\left(1-\frac{2c}{x}+\frac{a^2}{x^2}\right)$$

$$=x^2+2(a-c)x+2a(a-2c)+2a^2(a-c).\frac{1}{x}+\frac{a^4}{x^2}.$$

Also solved by G. B. M. Zerr.

CALCULUS.

286. Proposed by R. D. CARMICHAEL, Princeton University.

Solve the differential equation

$$[a_0x^3+a_1x^2y+a_2xy^2+(a_0-a_1+a_2)y^3 +a_3x^2+a_4xy+a_5y^2+a_6x+a_7y+a_8]dx +[a_0y^3+a_1xy^2+a_2x^2y+(a_0-a_1+a_2)x^3 +a_3y^2+a_4xy+a_5x^2+a_6y+a_7x+a_8]dy=0.$$

I. Solution by W. W. BEMAN, Prosessor of Mathematics, University of Michigan, Ann Arbor, Mich.

Putting $x=s+t$, $y=s-t$, and afterward $t^2=w$, the equation takes the linear form

$$\frac{dw}{ds}+\frac{4(3a_0-2a_1+a_2)s+2(a_3-a_4+a_5)}{4(a_1-a_2)s^2+2(a_3-a_5)s+a_6-a_7}.w$$

$$=-\frac{4(a_0+a_2)s^3+2(a_3+a_4+a_5)s^2+2(a_6+a_7)+2a_8}{4(a_1-a_2)s^2+2(a_3-a_5)s+a_6-a_7}.$$

Or, putting $x+y=2u$, $xy=2v$, the equation takes the linear form

$$\frac{dv}{du}+\frac{4(3a_0-2a_1+a_2)u+2(a_3-a_4+a_5)}{4(a_1-a_2)u^2+2(a_3-a_5)u+a_6-a_7}.v$$

$$=\frac{8a_0u^3+4a_3u^2+2a_6u+a_8}{4(a_1-a_2)u^2+2(a_3-a_5)u+a_6-a_7}.$$

From each of these solutions it is obvious that the original equation has an integrating factor of the form $f(x+y)$.

II. Solution by E. B. ESCOTT, Ann Arbor, Mich.

Call the given equation $Mdx+Ndy=0\ldots(2)$.
If F is an integrating factor, we must have

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=\frac{N\frac{\partial F}{\partial x}-M\frac{\partial F}{\partial y}}{F}\ldots(3).$$

From symmetry of M and N, we may suppose F to be symmetrical in x and y. Suppose F is a function of $x+y$. Then

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z},$$

and equation (3) becomes an ordinary differential equation instead of a partial differential equation. (3) becomes when the values of M and N are substituted,

$$(4a_1-3a_0-3a_2)(x^2-y^2)+(a_4-2a_5)(x-y)=$$

$$-[(a_1-a_2)(x+y)^2+(a_3-a_5)(x+y)+a_6-a_7](x-y)\frac{\frac{dF}{dz}}{F},$$

or, dividing by $x-y$ and putting $x+y=z$,

$$(4a_1-3a_0-3a_2)z+a_4-2a_5=-[(a_1-a_2)z^2+(a_3-a_5)z+a_6-a_7]\frac{dF}{dz}\cdot\frac{1}{F}.$$

The variables are separable, and we have

$$\frac{dF}{F}=-\frac{(4a_1-3a_0-a_2)z+a_4-2a_5}{(a_1-a_2)z^2+(a_3-a_5)z+a_6-a_7}dz=\left(\frac{a}{z-\alpha}+\frac{b}{z-\beta}\right)dz.$$

Then $F=(z-\alpha)^a(z-\beta)^b$, etc.

291. Proposed by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.

Integrate $\frac{dy}{dx}=ay^2+bx^m$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

This is the famous Riccati equation. It can be solved only under certain circumstances.

If $m=0$, we have $\frac{dy}{ay^2+b}+dx=0$, easily integrated. If $m=-2$, $\frac{dy}{dx}=ay^2+\frac{b}{x^2}$ becomes homogeneous, if y is replaced by z^{-1}. If $m=-4$, we put $y=\frac{1}{ax}+\frac{z}{x^2}$, then we have $\frac{dz}{az^2+b}+\frac{dx}{x^2}=0$, which can easily be integrated. When m has the form $-\frac{4n}{2n-1}$ or $-\frac{4n}{2n+1}$ the equation yields integrable forms, putting in the former case $x=1/u$, in the latter $y=1/z$.

This problem is fully treated in Johnson's *Differential Equations*, Chapter IX, and in Forsyth's *Differential Equations*, second edition, pages 170-176. A number of our contributors referred to these and other sources. Ed.F.

MECHANICS.

240. Proposed by S. A. COREY, Hiteman, Iowa.

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

Discussion by F. H. SAFFORD, Ph. D., University of Pennsylvania.

This problem involves several points of theoretical interest which may be better treated as a general problem. The analysis will be carried to such a stage that the transcendental equations involved may be used to obtain numerical results by any convenient method of computation.

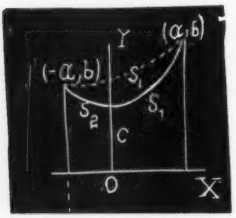
It is evident that between the supports the rope will hang in a catenary which, if referred to appropriate axes to be found, has the equation

$$y=c\cosh\frac{x}{c}. \qquad (1)$$

Let the tops of the two supports be $(-a, b)$ and (a_1, b_1), and let the free end of the rope hang from the first point.

"The tention at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of the point" (Minchin's *Statics, or* Bowser's *Analytic Mechanics*). With this use of the term "ordinate" the X-axis must be the directrix of the catenary, and since in this problem the rope is of unit length, weights and lengths are numeric-

ally the same. Thus b is the length of the free end in the initial position of the rope. It is desirable to express the constants in the problem in terms of quantities which from the point of view of Mechanics would naturally be given, *i. e.*, the difference in level of the supports, the perpendicular distance between them and the length of the rope in the catenary. Thus with the notation used above let us write



$$\begin{aligned} a+a_1&=2d,\\ b_1-b&=h,\\ s+s_1&=l, \end{aligned} \tag{2}$$

where s and s_1 are the lengths of the left and right portions of the catenary measured from the vertex. Since the curve passes through the points $(-a, b)$ and (a_1, b_1) it follows from (1) that

$$\begin{aligned} b&=c\cosh\left(\frac{-a}{c}\right)=c\cosh\frac{a}{c}\\ b_1&=c\cosh\frac{a_1}{c}. \end{aligned} \tag{3}$$

The values of s and s_1 are found to be

$$s=c\sinh\frac{a}{c},\quad s_1=c\sinh\frac{a_1}{c}. \tag{4}$$

From (2), (3), and (4),

$$\begin{aligned} h=b_1-b&=c\left[\cosh\frac{a_1}{c}-\cosh\frac{a}{c}\right]=2c\sinh\frac{a+a_1}{2c}\sinh\frac{a_1-a}{2c}\\ l=s+s_1&=c\left[\sinh\frac{a_1}{c}+\sinh\frac{a}{c}\right]=2c\sinh\frac{a+a_1}{2c}\cosh\frac{a_1-a}{2c}. \end{aligned} \tag{5}$$

$$\begin{aligned} h&=2c\sinh\frac{d}{c}\sinh\frac{a_1-a}{2c}\\ l&=2c\sinh\frac{d}{c}\cosh\frac{a_1-a}{2c}. \end{aligned} \tag{6}$$

$$\sqrt{(l^2-h^2)}=2c\sinh\frac{d}{c}, \tag{7}$$

$$a_1-a=2c\tanh^{-1}\frac{h}{l}. \tag{8}$$

Equation (7), though transcendental, is the most important result at this stage, as it may be shown that from it a single positive value of c is obtainable. From (8) and the first (2) followed by (3)

$$\begin{aligned} a&=d-c\tanh^{-1}\frac{h}{l},\\ a_1&=d+c\tanh^{-1}\frac{h}{l},\\ b&=c\cosh\left[\frac{d}{c}-\tanh^{-1}\frac{h}{l}\right],\\ b_1&=c\cosh\left[\frac{d}{c}+\tanh^{-1}\frac{h}{l}\right]. \end{aligned} \tag{9}$$

Thus from (7) and (9) the positions of the vertex and directrix of the catenary may be found. [In the original numerical problem $b=600$, $b_1=630$, $2d=300$, and the elimination of a and a_1 from (3) and the first of (2) gives

$$c\left[\cosh^{-1}\frac{b}{c}+\cosh^{-1}\frac{b_1}{c}\right]=2d. \tag{10}$$

The value of c must be found from (10) and then a and a_1 from (3), after which l follows from (7) and the solution may proceed as in the general case.]

The work performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope may be computed from the difference in level of the center of gravity of the entire rope before and after the change in position. Assuming that the rope will admit of pushing as well as pulling, the work may even be negative. In taking a and a_1 positive the lowest point is between the supports, but when a is 0 the vertex is at the top of the lowest support and there is no work to be done. When a is not 0 the problem is possible provided s_1 is less than the distance between the tops of the supports, *i. e.*,

$$c\sinh\frac{a_1}{c} < \sqrt{(h^2+4d^2)}. \tag{11}$$

This inequality is the closest criterion to be imposed upon the given quantities l, h, and d, others arising earlier being omitted because now superfluous. The ordinate of the center of gravity of the rope in the two positions involves the determination, for the catenary, of

$$\int y ds = c\int \cosh^2\frac{x}{c}dx = \frac{c^2}{4}\sinh\frac{2x}{c} + \frac{cx}{2}. \tag{12}$$

Between the limits $-a$ and a_1, (12) gives

$$\int_{-a}^{a_1} y ds = \frac{c^2}{2}\sinh\frac{2d}{c}\cosh 2\left(\tan^{-1}\frac{h}{l}\right) + cd$$

$$= \frac{c^2}{2}\cdot\frac{l^2+h^2}{l^2-h^2}\sinh\frac{2d}{c} + cd. \tag{13}$$

Taking into account the free end of the rope, the center of gravity of the entire rope in the initial position may now be found.

For the final state of the rope a new c must be computed in terms of h, d, and a new l which is the original s_1, [vide (4)], $\bar{c}$ being given by

$$\sqrt{(s_1^2-h^2)} = 2\bar{c}\sinh\frac{d}{\bar{c}}. \tag{7a}$$

The equation of the curve is now

$$y = \bar{c}\cosh\frac{x}{\bar{c}}, \tag{1a}$$

(referred to new axes), and the positions of the new vertex and directrix may be computed as before. The limits in (12) are to be taken as 0 and $2d$, c becoming $\bar{c}$, hence

$$\int_0^{2d} y ds = \frac{\bar{c}^2}{4}\sinh\frac{4d}{\bar{c}} + \bar{c}d. \tag{13a}$$

The free end is now longer by s, and the ordinate of the center of gravity of the new system may be found as before, though measured from the new directrix. The difference in level of the two centers of gravity may now be

readily obtained. In numerical problems the task of solving the transcendental equations (7) or (10), and (7a) presents no practical difficulty.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

172. Proposed by H. C. FEEMSTER, York, Neb.

Show that $\frac{(nr)!}{n!(r!)^n}$ is an integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

By induction we get the following:
Suppose $r(n-1)!$ is divisible by $(r!)^{n-1}(n-1)!$. Now

$$\frac{(rn)!}{n!(r!)^n} \div \frac{r(n-1)!}{(r!)^{n-1}(n-1!} = \frac{(nr)!}{r\ (n-1)!} \times \frac{(n-1)\ !}{n!r!}$$

$$= \frac{nr(nr-1)\,(nr-2)\ldots\text{to } r \text{ factors}}{nr(r-1)!}$$

$$= \frac{(nr-1)\,(nr-2)\ldots\text{to } (r-1) \text{ factors}}{(r-1)!} = \text{an integer.}$$

Now $\frac{r!}{r!\,1!}$ = an integer, and hence $\frac{(2r)!}{(r!)^2 2!}$ = an integer; $\frac{(3r)!}{(r!)^3 3!}$ = an integer, and so on.

173. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find integral values satisfying the equation,

$$a_1^2+a_2^2+a_3^2+\ldots+a_n^2=d^4.$$

I. Solution by FRANK LOXLEY GRIFFIN, Ph. D., Williams College.

One set of solutions may be obtained by putting $a_1=df_1$, $a_2=df_2$, etc., which reduces the problem to that of finding a sequence of n integers, the sum of whose squares is a perfect square. Or, geometrically, we seek $n-1$ right triangles whose sides are all integers, and the hypotenuse of each being one leg of the next.

I. This is readily accomplished for $n=2$ by recalling that, if (p^2-q^2) and $2pq$ are legs, where p and q are integers, the hypotenuse is also an integer, (p^2+q^2). Let f_1 be *any odd integer* (>1) and take $p-q=1$ and $p+q=f_1$, so that $p=\frac{1}{2}(f_1+1)$, $q=\frac{1}{2}(f_1-1)$, both integers. Thus the sides

of the right triangle are f_1, $\frac{1}{2}(f_1^2-1)$, and $\frac{1}{2}(f_1^2+1)$. Or, calling the second f_2 and the third d, we have $d^2=f_1^2+f_2^2$.

II. Let this value of d for $n=2$ be denoted by d_2, and observe that d_2 is *odd*. For $f_1=$say, $2m+1$ (being odd), whence $d_2=2m^2+2m+1$, odd. Thus d_2 may be treated precisely as was f_1, giving $f_3=\frac{1}{2}(d_2^2-1)$, $d_3=\frac{1}{2}(d_2^2+1)$, or $d_3^2=d_2^2+f_3^2=f_1^2+f_2^2+f_3^2$.

Since again d_3 is odd, the process may clearly be continued indefinitely. The successive values of f and d may be found from: $2f_{n+1}=d_n^2-1$, $2d_{n+1}=d_n^2+1$. The following table gives the values up to $n=6$ if we begin with f_1+3:

f_n	3	4	12	84	3612	6526884
d_n		5	13	85	3613	6526885

To obtain the proposed a's for any value of n, merely multiply each f (up to f_n) by d_n. Thus, for $n=3$,

$$39^2+52^2+156^2=13^4.$$

[We might also start with any *even* integer (>2), and either (A) form a sequence all terms of which are even multiples of the terms obtaining by beginning with any odd factor of f_1, or (B) let $f_1=2pq$ and take for p, q *any* factors whose product is $\frac{1}{2}f_1$. Thus for $f_1=14$:

$$p=7,\quad q=1,\quad f_2=48,\quad d_2=50,$$
$$p_2=25,\quad q_2=1,\quad f_3=624,\quad d_3=626,\text{ etc.}]$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Dr. Martin's solution can be extended to suit this requirement.

Let $x_1^2+x_2^2+x_3^2+\ldots+x_n^2=c^2$.

Then $c^2-n^2=(c-n)(c+n)=x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2$.

Let $c-n=b$, then $c+n=(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2)/b$.

$\therefore x_n=\frac{1}{2}(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2-b^2)/b$.

$c=\frac{1}{2}(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2+b^2)/b$.

Multiplying through by c^2 after substituting, we get

$$(cx_1)^2+(cx_2)^2+(cx_3)^2+\ldots+[(c/2b)(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2-b^2)]^2$$
$$=[(1/2b)(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2+b^2]^4.$$

$$\therefore (4b^2cx_1)^2+(4b^2cx_2)^2+(4b^2cx_3)^2+\ldots[2bc(x_1^2+x_2^2+x_3^2+\ldots$$
$$+x_{n-1}^2-b^2)]^2=(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2+b^2)^4.$$

Let $n=7$, $b=3$, x_1, x_2, x_3, $\ldots=1, 2, 3, \ldots, 6$. Then $c=\frac{50}{3}$.

$\therefore (600)^2+(1200)^2+(1800)^2+(2400)^2+(3000)^2+(3600)^2+(8200)^2=(100)^4$, or $6^2+12^2+18^2+24^2+30^2+36^2+82^2=10^4$.

Also solved by the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

240. Proposed by W. J. GREENSTREET, M. A., Editor The Mathematical Gazette, Stroud, England.

Let $S_{n-1}=1^{n-1}+2^{n-1}+3^{n-1}+\ldots+(n-1)^{n-1}$. Find n if $S_{n-1}-(n-1)$ is a multiple of n^2.

341. Proposed by O. L. CALLICOTT, Gettysburg, South Dakota.

Prove that the sum of the series, $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots$ to infinity = the sum of the series $\frac{1}{2}.1+\frac{1}{2^2}.\frac{1}{2}+\frac{1}{2^3}.\frac{1}{3}+\frac{1}{2^4}.\frac{1}{4}+\ldots$ to infinity.

342. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Prove that $\frac{1}{1.2.3.4}+\frac{1}{5.6.7.8}+\ldots=\frac{1}{4}\log 2-\frac{1}{24}\pi$. [Hobson's *Plane Trigonometry*, page 348.]

GEOMETRY.

371. Proposed by W. S. HUGHES, Student, Williams College.

A right circular cone is cut by two parallel planes, one passing through the vertex, and each cutting both nappes. Are the straight lines which constitute the first section parallel to the asymptotes of the hyperbola forming the other section?

372. Proposed by DANIEL KRETH, Oxford, Iowa.

In the right triangle ADE right angle A, are given: $AB=9$, $BC=280$, $CD=35$, angle AEB=angle CED; required the distance AE.

373. Proposed by S. LEFSEHETZ, East Pittsburg, Pa.

Draw a circle passing through a given point and orthogonal to two given circles.

CALCULUS.

297. Proposed by PROF. L. C. WALKER, Socorro, New Mexico.

A square hole $2s$ on a side is cut through an ellipsoid, axes $2a$, $2b$, $2c$, the axis of the hole coinciding with the axis $2c$ of the ellipsoid. Find (1) the volume, and (2) the surface removed.

298. Proposed by C. N. SCHMALL, New York City.

Prove, by calculus, that if two regular polygons have equal perimeters, that which has the greater number of sides has the greater area.

399. Proposed by JOSEPH V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

A cow is pasturing outside a circular field containing 10 acres. What length of rope will allow her to graze over exactly two acres?

MECHANICS.

351. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

ABC is a uniform triangular lamina of weight $3W$ such that $AB=2AC$. A particle of weight W is attached to it at C. Show that if the lamina be suspended from angle A, it will rest with AB and AC equally inclined to the vertical.

352. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Assuming the earth's orbit to be a circle, if a comet move in an ellipse around the sun, in the same plane as the earth, with its perihelion nearer the sun than the earth, how long will it remain within the earth's orbit? Apply this to Halley's comet with the above assumptions, and supposing the eccentricity of its orbit to be 0.971733. Find its periodic time.

353. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

175. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Nebraska.

Show (a) that $[(2n)!]/[(n+1)!n!]$ is an integer, and (b) that $[(2a)!(2b)!]/[a!b!(a+b)!]$ is an integer.

AVERAGE AND PROBABILITY.

206. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A point O is taken at random in a triangle. What is the probability that if three other points are taken at random, one shall lie in each of the three triangles AOB, BOC, and COA? [Williamson's *Integral Calculus*, page 412, ex. 85.]

207. Proposed by JOSEPH V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

A formula for simple interest is, $i=prt/12$. If r is 5 or 7, what is the chance that the denominator will cancel out entirely into either p or t or both (where t is expressed in months and tenths of a month, written decimally with usually a common fraction after tenths). It is assumed that a factor of 12 can cancel into t if the integral part is divisible by 2, 3, or 4. Thus, to find the interest at 5% on \$729.44 for 1 year, 9 months, 28 days, we write

$$1.8236\times5\times\frac{21.9\frac{1}{3}}{12}=?$$

NOTES AND NEWS.

The August-September number of the MONTHLY will be published September 28th.

At the University of Pennsylvania hereafter the chairman of each department of instruction will be elected annually by the department. For the year 1910-1911, Professor George Egbert Fisher has been chosen chairman of the Department of Mathematics in the Graduate School, and Professor Isaac J. Schwatt chairman of the Mathematical Deparmennt in the College.

In the Department of Mathematics at the University of Pennsylvania Assistant Professor G. H. Hallett has been promoted to a professorship and Dr. M. J. Babb, Dr. G. G. Chambers and Dr. O. E. Glenn have been promoted to instructorships.

At Purdue University Professor Erastus Test will retire at the close of the present academic year, and Professor Jacob Westlund has been promoted to a full professorship in mathematics.

The series of articles on the teaching of collegiate mathematics, which was closed for the present year with the paper by Professor Bailey in the last issue, is appropriately supplemented in this number by Professor Miller's paper on "Mathematics Beyond the Calculus," which is intended primarily for students but which should prove equally valuable to many teachers. There are already indications that this series of papers will bear fruit in further discussions, through these columns, of many important phases of the educational side of mathematics, which deserves careful reconsideration.

Mr. T. H. Hildebrandt, who is instructor in mathematics at the University of Michigan, has just completed his work for the Doctorate at the University of Chicago by presenting his thesis on "A contribution to the Foundations of Frechet's Calcul Fonctionnel." The degree was conferred at the convocation on June 14, 1910.

"The Absolute Minimum of a Definite Integral in a Special Field" is the title of the thesis presented by Mr. E. J. Miles for the Doctorate at the University of Chicago. Dr. Miles has been appointed to an instructorship at Cornell University.

Miss Marion B. White has received an appointment as assistant professor of mathematics at the University of Kansas. Miss White was formerly instructor at the University of Illinois, which position she left to pursue graduate work at the University of Chicago.

Dr. Elizabeth R. Bennett has been appointed instructor in mathematics at the University of Nebraska, and not at the University of Kansas as previously announced in these columns.

Mr. J. O. Pyle, professor of mathematics at Howard Payne College, Texas, has resigned to pursue advanced work at the University of Chicago.

Dr. R. C. Archibald, of Brown University, has spent the year 1909-10, on leave of absence, studying in Paris. He returns next year as assistant professor of mathematics.

Mr. H. C. Currier, instructor in mathematics at Brown University, has leave of absence for the coming academic year. He will spend the year in study in German universities.

Dr. Phillips, instructor in mathematics at the Massachusetts Institute of Technology, is attending lectures at the University of Chicago during the present Summer.

Professor F. S. Woods, of the Massachusetts Institute of Technology, will resume his duties in the autumn, after a year's absence in Europe.

Mr. C. J. West, fellow in mathematics in the University of Illinois, has been appointed instructor in mathematics in the University of Ohio.

Mr. Benjamin Carter, of the Massachusetts Institute of Technology, has been appointed instructor in mathema,ics at Colby College, Maine.

Professor G. A. Miller will teach in the summer session at the University of Illinois.

Dr. O. D. Kellogg, of the University of Missouri, has been promoted to a full fellowship in mathematics.

Mr. R. P. Baker, instructor in mathematics at the University of Iowa, has been promoted to an assistant professorship.

Professor E. H. Moore will spend six weeks in Colorado and return to lecture at the University of Chicago during the second term of the summer quarter.

BOOKS.

First Course in Algebra. By Herbert E. Hawkes, Assistant Professor of Mathematics in Yale University, William A. Luby, and Frank C. Touton, Instructors in Mathematics, Central High School, Kansas City, Mo. 12mo, cloth, vii+334 pages, list price $1.00. Boston and Chicago: Ginn & Co.

"The 'First Course in Algebra' is designed for the first year's work. The topics considered have been strictly limited to those which belong primarily to study in the first year. Many helpful suggestions, the fruit of the widespread discussion of mathematical teaching which has marked the progress of the past ten years, are embodied in the book.

Difficult exercises have been avoided. Those given are new, varied, graded with extreme care, and amply sufficient to develop the essentials of elementary technic. The principles, so far as possible, are developed from the student's knowledge of arithmetic.

An abundance of typical solutions are given, and rules based on them have been carefully stated in full. Wherever practicable, suitable methods of checking are illustrated."

Plain Trigonometry. By Fletcher Durel, Ph. D., Head of the Mathematics Department of the Lawrenceville School. 8vo, cloth. 184+114 pages of tables. Price, $1.25. New York: Charles E. Merrell Co.

Among the special features claimed for this book, are the following: Under each case in the solution of triangles, examples are given in which the degrees are divided sexagessimally and decimally, the latter division of the degree meeting the new requirements of Harvard, Yale, and Princeton. A chapter given to the treatment of logarithms and other properties; a chapter in which the application of trigonometry are reduced to a system; and the enlivening of the subject matter by a chapter on the history of Trigonometry.

The author, in the arrangement of his logarithmic work adopted the form of tabulation used in the U. S. Navy Department and by engineers in general. This, to my mind, is of doubtful pedagogical value, since it omits the equality signs and thus breaks the solution into fragments instead of presenting it in concise symbolic sentences.

The book is neatly written and systematically arranged. The publishers have done their part to make the work attractive to both teacher and student. F.

Factor Table for the First Ten Millions. By D. N. Lehmer, Washington, D. C., Carnegie Institute of Washington, 1909. xiv+476 pages.

The chief factor tables published hitherto are those by Burckhardt for the first three millions, Glaisher for the next three millions, and Dase and Rosenberg for the seventh, eighth and ninth millions; they contain no less than 246 errors listed by Lehmer. In view of the difficulty of obtaining copies of certain of these tables, the appearance of Lehmer's table in a single volume will be very welcome to arithmeticians. The greater accuracy of Lehmer's tables rests not merely upon the essential improvements in its construction but chiefly upon the fact that it was constructed independently of the earlier tables, so that errors could be eliminated by noting discrepancies with the earlier tables, as detected by a comparison, entry for entry, made five times. Finally a very important check was made by counting the primes within various limits and comparing these counts with the numbers computed by Bertelsen by a modification of Meissel's method, the results tallying in every case. We congratulate Dr. Lehmer on this happy conclusion of his formidable task and express our gratitude to the Carnegie Institution for its aid in the preparation and publication of this monumental table. L. E. DICKSON.

Elementary Experimental Mechanics. By A. Wilmer Duff, M. A., D. Sc., (Edinburgh) Professor of Physics in the Worcester Polytechnic Institute, Worcester, Mass. 12mo. Red cloth, ix+267 pages. Price, $1.60 net. New York: The Macmillan Co.

This book is an attempt to make mechanics as attratctive as possible by a suitable collection of laboratory exercises. Among them are the Addition of Displacements, Path of Projectile, Motion of Pendulum, Simple Harmonic Motion, Force and Acceleration, Conical Pendulum, Friction, Moment of Force, Kinetic and Potential Energy, Torsion Pendulum, the Gyroscope, Torsion of a Wire, Flexure of a Bar, Archimedes' Principle, etc., etc. The apparatus to cary out the experiments is of the simplest design. The explanation and discussion is clear and fairly exhaustive. The author tells us that while the exercises have been chosen with a view to elucidate principles, yet the necessity of precise measurement has been kept in mind. The aim of the course, he says, is to stimulate reflection on concepts and principles and the value of each exercise is in proportion to the importance and number of physical ideas which must be considered in performing the exercise.

The book is neat and attractive in appearance, and its contents will be very valuable to those teachers who can avail themselves of its use. F.

Elementary Projective Geometry. By A. G. Pickford, Sometime Scholar of St. John's College, Cambridge, Headmaster of the Holme Grammar School, Oldham. 8vo. Red cloth, xi+256 pages. Price, 4s. Cambridge, Eng.: The University Press. New York. U. S. A.: G. P. Putnam's Sons.

The author of this work has arranged in orderly sequence the elementary propositions of Plane Projective Geometry, assuming on the part of the student only a knowledge of the first six books of Euclid or their equivalent. Proceeeing from the projective unit, the cross-ratio of four colinear points or four concurrent lines, the author takes up the discussion of projective rows and pencils, and the involution of six points or lines. He then deduces the properties of the curves of the second degree, the properties of the polars, and inscribed and circumscribed polygons, with construction of conics satisfying five given conditions.

The book ends with a brief consideration of Polar Reciprocation and Plane Homology, also brief notes on Projection in Space. In keeping with all work done by the University Press, the typography and mechanical execution of the book is first class. F.

An Introduction to Physical Science. By Frederick H. Getman, Ph. D., Associate in Chemistry, Bryn Mawr College, formerly Carnegie Research Assistant in Physical Chemistry, Johns Hopkins University. 12mo, cloth. v+257 pp. 129 figures. Price, $1.50 net. New York: John Wiley & Sons.

The author of this work, believing that many of the difficulties encountered by the beginner in chemistry may be traced to his unfamiliarity with the metric system, with the laws of elementary mechanics, or with simple thermal and electrical phenomena, has prepared this book to meet the needs of this class of students.

The work thus embodies a series of simple experiments demonstrating the more important phenomena in heat, light, electricity, and magnetism. F.

An Introduction to the Science of Radio-Activity. By Charles W. Raffety. With Illustrations. 8vo, cloth, xii+208 pages. Price, $1.25 net. New York: Longmans, Green & Co.

This book attempts "to give a concise and popular account of the properties of the radio-active elements and the theoretical conceptions which are introduced by the study of radio-active phenomena." The author has succeeded very well in giving the reader a fair idea of the main discoveries made during the last decade in this most interesting field of physical research.

The book is divided into three parts: The descriptive, containing eight chapters; the theoretical, containing eight chapters; and the practical, containing five chapters.

Every chapter is full of interest. F.

Problemes et Exercises de Mathématiques Générales. E. Fabry, Professeur a L'Universite de Montpellier. 8vo. Paper, 420, pages. Price, 10fr. Paris: A. Hermann & Sons.

In this book is to be found a fine collection of problems in Algebra, Analytic Geometry, Analysis, and Mechanics, covering the first 80 pages. The remainder of the book is devoted to the solution of the problems and their discussion. It is a good book from which to draw problems for supplementary exercises. F.

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ON THE CLASSIFICATION OF SYSTEMS OF LINEAR EQUATIONS.

By ARTHUR RANUM, Cornell University.

In a recent paper in THE AMERICAN MATHEMATICAL MONTHLY,* "On the Solution of a System of Linear Equations," Professor Miller has proved that in all the solutions of a system of consistent linear equations a particular unknown x_i will have the same value, that is, will satisfy an equation of the form $bx_i+l=0$ $(b\neq 0)$, if and only if the omission of the coefficients of x_i from the matrix of the system reduces the rank of the matrix. This theorem immediately suggested to me a generalization in which several unknowns are involved instead of only one, and I soon found that the more general theorem so derived is a sufficient basis for, and naturally gives rise to, a complete classification of systems of consistent linear equations, not only with respect to the rank of their matrices, but also with respect to the rank of the partial matrices obtained by omitting the coefficients of one or more unknowns. In this note I wish to state and prove the theorem, explain the resulting classification, apply it to one or two simple cases, and point out its geometrical significance.

REDUCTION IN RANK.

1. Consider any system T of m linear equations in $n=s+t$ unknowns, as follows:

$$T\begin{cases} a_{11}x_1+\ldots+a_{1s}x_s+a_{1,\,s+1}x_{s+1}+\ldots+a_{1n}x_n+k_1=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1+\ldots+a_{ms}x_s+a_{m,\,s+1}x_{s+1}+\ldots+a_{mn}x_n+k_m=0. \end{cases}$$

Theorem. *A necessary and sufficient condition that in all the solutions of a system of consistent linear equations the values of any particular set of unknowns x_{s+1}, ..., x_n, t in number, are connected by the same linear relation of the form*

*Vol. XVII (1910), p. 137.